PART 1: QUESTIONS

Name:________________________________ **Age:**_____ **Id:**_______________ **Course:**__________

Derivatives - Exam 2 **Lesson: 4-6**

Instructions:

- Please begin by printing your Name, your Age, your Student Id , and your Course Name in the box above and in the box on the solution sheet.
- You have 90 minutes (class period) for this exam.
- You can not use any calculator, computer, cellphone, or other assistance device on this exam. However, you can set our flag to ask permission to consult your own one two-sided-sheet notes at any point during the exam (You can write concepts, formulas, properties, and procedures, but questions and their solutions from books or previous exams are not allowed in your notes).
- Each multiple-choice question is worth 5 points and each extra essay-question is worth from 0 to 5 points. (Even a simple related formula can worth some points).
- Set up your flag if you have a question.
- Relax and use strategies to improve your performance.

Exam Strategies to get the best performance:

- Spend 5 minutes reading your exam. Use this time to classify each Question in (E) Easy, (M) Medium, and (D) Difficult.
- Be confident by solving the easy questions first then the medium questions.
- Be sure to check each solution. In average, you only need 30 seconds to test it. (Use good sense).
- Don't waste too much time on a question even if you know how to solve it. Instead, skip the question and put a circle around the problem number to work on it later. In average, the easy and medium questions take up half of the exam time.
- Solving the all of the easy and medium question will already guarantee a minimum grade. Now, you are much more confident and motivated to solve the difficult or skipped questions.
- Be patient and try not to leave the exam early. Use the remaining time to double check your solutions.

1. Given the function $y = x^2 - 5x + 6$. Then:

I. It is an example of **Explicit Function** in which the dependent variable y is written **explicitly** in terms of the independent variable x.

II. x is the dependent variable and is given in terms of the independent variable y.

III. It is an example of **Implicit function** in which the dependent variable y has not been given **explicitly** in terms of the independent variable x.

a) Only I is correct.
b) Only II is correct.
c) Only III is correct.
d) Only I and II are correct.
e) Only II and III are correct.

Solution: a

I. True. $y = x^2 - 5x + 6$ is an explicit function in which the dependent variable y is expressed in terms of the independent variable x.

II. False. y is the dependent variable and is given in terms of the independent variable x.

III. False. $y = x^2 - 5x + 6$ is not an example of **Implicit function** in which the dependent variable y has not been given "explicitly" in terms of the independent variable x.

Thus, only I is correct.

2. Let y be an implicit function on x in the equation:

$$e^{(x+y)} = \tan(x - y^2), \text{ where } y = f(x).$$

Then:

I. It is easier to find $\frac{dy}{dx}$ by implicit derivative because it is difficult to find the explicit function $y = f(x)$.

II. It is easier to find $\frac{dy}{dx}$ by implicit derivative by differentiating each term in turn in both sides of the equation.

III. It is difficult to find $\frac{dy}{dx}$ by implicit derivative because all derivative rules such as: chain rule, product rule, quotient rule can only be applied on explicit functions.

a) Only I is correct.
b) Only II is correct.
c) Only III is correct.
d) Only I and II are correct.
e) I, II, and III are correct.

Solution: d

Some applications, the variables are related by an equation rather than a function.

In these cases, you can find the derivative of one variable with respect to the other by the implicit derivative.

In the equation $e^{(x+y)} = \tan(x - y^2)$, where $y = f(x)$, it is easier to find $\frac{dy}{dx}$ by implicit derivative because it is difficult to find the explicit function $y = f(x)$.

The implicit derivative consists in differentiating each term in turn in both sides of the equation by using the chain rule or any other derivative rule such that: product rule, quotient rule, etc..

3. Let y be a function on x.

The implicit derivative $\frac{dy}{dx}$ of the equation $x^4 = y^3$ is:

a) $\frac{dy}{dx} = \frac{x}{y}$

b) $\frac{dy}{dx} = \frac{3x^2}{2y}$

c) $\frac{dy}{dx} = \frac{4x^3}{3y^2}$

d) $\frac{dy}{dx} = \frac{5x^4}{7y^6}$

e) None of the above.

Solution: c

Let $x^4 = y^3$. To find $\frac{dy}{dx}$, we can differentiate both sides of the equation with respect to x.

$$\frac{d}{dx}\left(x^4\right) = \frac{d}{dx}\left(y^3\right)$$

$$4x^3 = 3y^2\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4x^3}{3y^2}.$$

4. Let y be a function on x such that $y^4 = 3xy$.

The derivative $\frac{dy}{dx}$ at $P(1,1)$ is:

a) $\frac{dy}{dx} = 1$

b) $\frac{dy}{dx} = 2$

c) $\frac{dy}{dx} = 3$

d) $\frac{dy}{dx} = 4$

e) None of the above.

Solution: c

The implicit derivative consists in differentiating each term in turn in both sides of the equation $y^4 = 3xy$ at $P(1,1)$ by using the Chain Rule, the Power Rule, and Product Rule.

The implicit derivative of $y^4 = 3xy$ at $P(1,1)$ is:

$$\frac{d}{dx}\left(y^4\right) = \frac{d}{dx}\left(3xy\right)$$

$4y^3\frac{dy}{dx} = 3.y + 3x.1.\frac{dy}{dx}$. At $P(1,1)$, we have:

$$4\frac{dy}{dx} = 3 + 3\frac{dy}{dx}$$

$$\frac{dy}{dx} = 3.$$

5. Let x and y be functions on t.

If $x + y = t^3$ then $\frac{dx}{dt} + \frac{dy}{dt}$ is:

a) $\frac{dx}{dt} + \frac{dy}{dt} = 0$

b) $\frac{dx}{dt} + \frac{dy}{dt} = 1$

c) $\frac{dx}{dt} + \frac{dy}{dt} = 2t$

d) $\frac{dx}{dt} + \frac{dy}{dt} = 3t^2$

e) None of the above.

Solution: d

Let $x + y = t^2$.

We can find $\frac{dx}{dt} + \frac{dy}{dt}$ by implicit derivative that consists in differentiating on t both sides of the equation.

$$\frac{d}{dx}(x+y) = \frac{d}{dx}(t^3)$$

$$\frac{dx}{dt} + \frac{dy}{dt} = 3t^2.$$

6. Let x and y be functions on t.

Find $\frac{dy}{dt}$ at $P(1,1)$ of the equation $x^6 - y^5 = 88$.
Given $\frac{dx}{dt} = 1$ at $P(1,1)$.

a) $\frac{dy}{dx} = \frac{3}{2}$
b) $\frac{dy}{dx} = \frac{4}{3}$
c) $\frac{dy}{dx} = \frac{5}{4}$
d) $\frac{dy}{dx} = \frac{6}{5}$
e) None of the above.

Solution: d

The implicit derivative consists in differentiating on t both sides of the equation.

$$\frac{d}{dt}(x^6 - y^5) = \frac{d}{dt}(88)$$

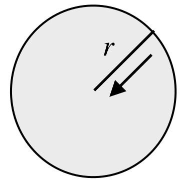
$$6x^5\frac{dx}{dt} - 5.y^4\frac{dy}{dt} = 0.$$

For $P(1,1)$ and $\frac{dx}{dt} = 1$, we have:

$$6(1)^5(1) - 5.(1)^4\frac{dy}{dt} = 0.$$

$$\frac{dy}{dx} = \frac{6}{5}.$$

7. A snowball is melting at a rate of 4 cubic inch per minute. At what rate is the radius changing when the snowball has a radius of 1 inches?



$$V_{sphere} = \frac{4\pi r^3}{3}$$

a) $\frac{dr}{dt} = -\frac{2}{\pi}\ \frac{\text{in}}{\text{min}}$

b) $\frac{dr}{dt} = -\frac{1}{\pi}\ \frac{\text{in}}{\text{min}}$

c) $\frac{dr}{dt} = \frac{1}{\pi}\ \frac{\text{in}}{\text{min}}$

d) $\frac{dr}{dt} = \frac{2}{\pi}\ \frac{\text{in}}{\text{min}}$

e) None of the above.

Solution: b

Let $r =$ radius, $V =$ volume.

We have $\frac{dv}{dt} = -4\frac{\text{in}^3}{\text{min}}$ (since volume is decreasing).

We want to find $\frac{dr}{dt} = ?$ when $r = 1$ in.

$$V = \frac{4\pi r^3}{3}$$

Differentiate with respect to time in both sides of the equation, we have:

$$\frac{dV}{dt} = \frac{4}{3}\pi 3r^2\frac{dr}{dt}.$$

Since $\frac{dv}{dt} = -4\frac{\text{in}^3}{\text{min}}$ and $r = 1$ in, then:

$(-4) = \frac{4}{3}\pi 3(1)^2 \frac{dr}{dt}$

Thus, $\frac{dr}{dt} = -\frac{1}{\pi} \frac{\text{in}}{\text{min}}$.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function on x and $f'(x)$ and $f''(x)$ be its first and its second derivatives.

I. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x)$ has a local maximum at $x_0 \in \mathbb{R}$.

II. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x)$ has a local minimum at $x_0 \in \mathbb{R}$.

III. If $f'(x_0) = 0$ and $f''(x_0) < 0$, then $f(x)$ has a local minimum at $x_0 \in \mathbb{R}$.

IV. If $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x)$ has a local maximum at $x_0 \in \mathbb{R}$.

Then:

a) Only I and II are correct.
b) Only II and III are correct.
c) Only III and IV are correct.
d) Only I and IV are correct..
e) None of the above.

Solution: a

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function on x and $f'(x)$ and $f''(x)$ be its first and its second derivatives then:

- $f(x)$ has a local maximum at $x_0 \in \mathbb{R}$ if $f'(x_0) = 0$ and $f''(x_0) < 0$.
- $f(x)$ has a local minimum at $x_0 \in \mathbb{R}$ if $f'(x_0) = 0$ and $f''(x_0) > 0$.

Thus, only I and II are correct.

9. Find the vertex $V(x_v, y_v)$ of the following function:

$y = x^2 - 2x + 2$

a) $V(1,1)$ is a maximum.

b) $V(1,1)$ is a minimum.

c) $V(1,-1)$ is a maximum.

d) $V(1,-1)$ is a minimum.

e) None of the above.

Solution: d

$f(x)$ has a local maximum at x_0 if $f'(x_v) = 0$ and $f''(x_v) < 0$.

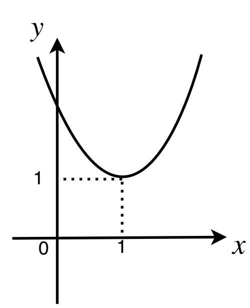
$f(x)$ has a local minimum at x_0 if $f'(x_v) = 0$ and $f''(x_v) > 0$.

$y = x^2 - 2x + 2$
$y' = 2x - 2$
$y'' = 2$

$y' = 0 \Rightarrow 2x_v - 2 = 0 \Rightarrow x_v = 1$
$y_v = x_v^2 - 2x_v + 2$
$y_v = (1)^2 - 2(1) + 2 \Rightarrow y_v = 1$

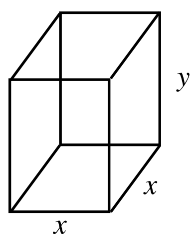
Since $y'' = 2 > 0$

Thus, $V(1,-1)$ is a minimum.



10. A rectangular box with a square bottom and no top is to have a volume of 1 cubic feet. The material for the bottom costs \$2 per square foot, while the material for the sides costs \$1 per square foot.

Find the dimensions of the box having the smallest possible cost.



a) $x_m = 1$ ft and $y_m = 1$ ft$x_m = 1$ ft and $y_m = 1$ ft.

b) $x_m = \frac{1}{2}$ ft and $y_m = 4$ ft.

c) $x_m = \frac{1}{4}$ ft and $y_m = 16$ ft.

d) $x_m = \sqrt{2}$ ft and $y_m = \frac{1}{2}$ ft.

e) None of the above.

Solution: a

The volume is:

$$V = x^2 y = 1 \text{ ft}^3 \Rightarrow y = \frac{1}{x^2}$$

The cost is:

$$C = (C_{bottom})(A_{bottom}) + (C_{sides})(A_{sides})$$

$$C = (2)(x^2) + (4)(xy)$$

$$C = 2x^2 + 4x\left(\frac{1}{x^2}\right)$$

$$C = 2x^2 + \frac{4}{x}$$

$$C' = 4x_m - \frac{4}{x_m^2} = 0 \Rightarrow \frac{4x_m^3 - 4}{x_m^2} = 0$$

$$4x_m^3 - 4 = 0 \Rightarrow x_m^3 = 1 \Rightarrow x_m = 1 \text{ ft}$$

$$y_m = \frac{1}{x_m^2} \Rightarrow y_m = \frac{1}{(1)^2} \Rightarrow y_m = 1 \text{ ft}$$

11. Let x and y be sides of a rectangle.

What is the maximum Area $A = xy$ such that the Perimeter ($P = 2x + 2y$) is 200 ft.

a) $A_{max} = 2{,}500$ ft²

b) $A_{max} = 3{,}000$ ft²

c) $A_{max} = 3{,}500$ ft²

d) $A_{max} = 4{,}000$ ft²

e) None of the above.

Solution: a

$$A = xy$$

$$P = 2x + 2y = 200 \Rightarrow x + y = 100 \Rightarrow y = 100 - x$$

Then,

$$P = xy \Rightarrow P = x(100 - x) \Rightarrow P = 100x - x^2$$

Maximizing the Area:

$$\frac{dP}{dx} = 0 \Rightarrow 100 - 2x_{max} = 0 \Rightarrow x_{max} = 50 \text{ ft}$$

$$y_{max} = 100 - x_{max} \Rightarrow y_{max} = 100 - 50$$

$$y_{max} = 50 \text{ ft}$$

$$A_{max} = x_{max} y_{max} \Rightarrow A_{max} = 2{,}500 \text{ ft}^2$$

12. For which values of x the function $y = \frac{x^3}{3} - 4x$ has a local maximum or minimum.

I. $x = 2$ (local minimum)

II. $x = -2$ (local minimum)

III. $x = 2$ (local maximum)

IV. $x = -2$ (local maximum)

Then:

a) Only II and III are correct.
b) Only I and II are correct.
c) Only III and IV are correct.
d) Only I and IV are correct.
e) None of the above.

Solution: b

$y = \frac{x^3}{3} - 4x$

The derivatives of y are:
$y' = x^2 - 4$
$y'' = 2x$

The local maximum or local minimum of y are:
$y' = 0 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$

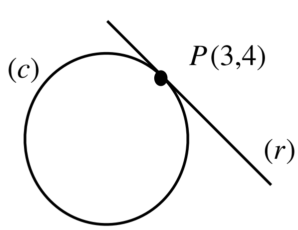
For $x = 2$ and $y''(1) = 2(2) = 4 > 0$ (local minimum)

For $x = -2$ and $y''(-2) = 2(-2) = -4 < 0$ (local maximum).

13. Find the equation of the tangent line (r) to the circle (c) $x^2 + y^2 = 25$ at the point $P(4,3)$.

Given: $y - y_0 = \frac{dy}{dx}(x - x_0)$.

Hint: Use implicit derivative to find the slope.



a) $y - 3 = -\frac{4}{3}(x - 4)$

b) $y - 4 = -\frac{3}{4}(x - 3)$

c) $y - 4 = \frac{3}{4}(x + 3)$

d) $y + 3 = \frac{4}{3}(x - 4)$

e) None of the above.

Solution: a

Using implicit derivative on x to find the slope $\frac{dy}{dx}$ on $P(4,3)$.

$x^2 + y^2 = 25 \Rightarrow 2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$.

At the point P, the slope is $\frac{dy}{dx} = -\frac{4}{3}$.

$y - y_0 = \frac{dy}{dx}(x - x_0) \Rightarrow y - 3 = -\frac{4}{3}(x - 4)$.

14. The increasing/Decreasing Test states that:

a) If $f''(x) > 0$ on an interval, then f is increasing on that interval and if $f''(x) < 0$ on an interval, then f is decreasing on that interval.

b) If $f''(x) < 0$ on an interval, then f is increasing on that interval and if $f''(x) > 0$ on an interval, then f is decreasing on that interval.

c) If $f'(x) > 0$ on an interval, then f is increasing on that interval and if $f'(x) < 0$ on an interval, then f is decreasing on that interval.

d) If $f'(x) < 0$ on an interval, then f is increasing on that interval and if $f'(x) > 0$ on an interval, then f is decreasing on that interval.

e) None of the above.

Solution: c

The **increasing/Decreasing Test** states that:

- If $f'(x) > 0$ on an interval, then f is increasing on that interval.
-

- if $f'(x) < 0$ on an interval, then f is decreasing on that interval.

15. The **Concavity Test** states that:

a) If $f''(x) > 0$ on an interval, then f is concave up on that interval and if $f''(x) < 0$ on an interval, then f is concave down on that interval.

b) If $f''(x) < 0$ on an interval, then f is concave up on that interval and if $f''(x) > 0$ on an interval, then f is concave down on that interval.

c) If $f'(x) < 0$ on an interval, then f is concave up on that interval and if $f'(x) > 0$ on an interval, then f is concave down on that interval.

d) If $f'(x) > 0$ on an interval, then f is concave up on that interval and if $f'(x) < 0$ on an interval, then f is concave down on that interval.

e) None of the above.

Solution: a

The **Concavity Test** states that:

- If $f''(x) > 0$ on an interval, then f is concave up on that interval.
- If $f''(x) < 0$ on an interval, then f is concave down on that interval.

16. The condition to find the inflection points of $f(x)$ is:

a) $f(x) = 0$
b) $f'(x) = 0$
c) $f''(x) = 0$
d) $f'''(x) = 0$
e) None of the above.

Solution: c

The condition to find the inflection points of $f(x)$ is $f''(x) = 0$.

17. Find x-intercept (x_{int}) and y-intercept (x_{int}) of $y = x^2 - 4$.

a) $x_{int} = \pm 2$ and $y_{int} = 4$
b) $x_{int} = \pm 2$ and $y_{int} = -4$
c) $x_{int} = -2$ and $y_{int} = \pm 4$
d) $x_{int} = 2$ and $y_{int} = \pm 4$
e) None of the above.

Solution: b

Let $y = x^2 - 4$.

The x-intercept (x_{int}) occurs when $y = 0$ then:

$$0 = x_{int}^2 - 4 \Rightarrow (x_{int} + 2)(x_{int} - 2) = 0 \Rightarrow x_{int} = \pm 2$$

The y-intercept (y_{int}) occurs when $x = 0$ then:

$$y_{int} = (0)^2 - 4 \Rightarrow y_{int} = -4.$$

18. Let $y = \dfrac{x^3}{3} - 36x$. The intervals where y is increasing are:

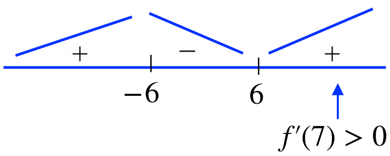
a) $x < -1$ or $x > 1$
b) $-1 < x < 1$
c) $-4 < x < 4$
d) $x < -16$ or $x > 16$
e) None of the above.

Solution: e

$$y = \frac{x^3}{3} - 36x$$
$$y' = x^2 - 36$$

The increasing/decreasing sign changes when $y' = 0$.

Then: $y' = x^2 - 36 = 0 \Rightarrow x = \pm 6$.



Thus, $x < -6$ or $x > 6$.

19. Let $y = \dfrac{x^3}{3} - 16x$.

The intervals where y is concave up are:

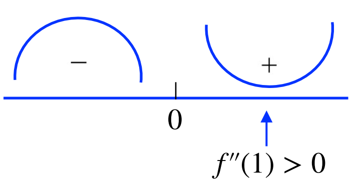
a) $x < 1$
b) $x < 0$
c) $x > 1$
d) $x > 0$
e) None of the above.

Solution: d

$$y = \frac{x^3}{3} - 16x$$
$$y' = x^2 - 16$$
$$y'' = 2x$$

The concavity sign changes when $y'' = 0$. Then:

$y'' = 2x \Rightarrow x = 0$.



Thus, $x > 0$.

20. Let $y = \dfrac{x^3}{3} - 16x$.

The asymptotes are:

a) $x = 0$ or $y = 1$
b) $x = 1$ or $y = 0$
c) $x = -1$ or $y = 1$
d) $\nexists$ Asymptotes.
e) None of the above.

Solution: d

- $\nexists$ Vertical Asymptotes because the denominator $D \neq 0$.
- $\nexists$ Horizontal Asymptotes because:

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} \frac{x^3}{3} - 16x = \infty$$
$$\lim_{x\to-\infty} y(x) = \lim_{x\to-\infty} \frac{x^3}{3} - 16x = -\infty.$$

PART 2: SOLUTIONS

Consulting

Name:______________________________ **Age:**_____ **Id:**________________ **Course:**__________

Multiple-Choice Answers

Questions	A	B	C	D	E
1					
2					
3					
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	Points	Max
Multiple Choice		100
Extra Points		25
Consulting		10
Age Points		25
Total Performance		160
Grade		A

Extra Questions

21. Given $yx + y = 11$, where y is a function on x.

Find the derivative $\dfrac{dy}{dx}$.

- **Method 1: Explicit Derivative**

  Step 1: Isolate y,

  Step 2: Find $\dfrac{dy}{dx}$ by differentiating explicitly y.

- **Method 2: Implicit Derivative**

  Step 1: Differentiate implicit on x the equation.

Show that $\dfrac{dy}{dx}$ is the same in the two methods to receive an extra 5 points.

Solution: $y' = -\dfrac{11}{(x+1)^2}$

- Method 1: Explicit Derivative

  Isolate y:

$$yx + y = 11 \Rightarrow y = \frac{11}{x+1}$$

Find $\dfrac{dy}{dx}$ by differentiating explicitly y.

$$y' = \frac{(0).(x+1) - (11)(1)}{(x+1)^2} \Rightarrow y' = -\frac{11}{(x+1)^2}$$

- Method 2: Implicit Derivative

  Differentiate implicit on x the equation.

$$yx + y = 11$$
$$y'x + y + y' = 0 \Rightarrow y' = -\frac{y}{x+1}$$

Since $y = \dfrac{11}{x+1}$ then:

$$y' = -\frac{y}{x+1} \Rightarrow y' = -\frac{\left(\frac{11}{x+1}\right)}{x+1} \Rightarrow y' = -\frac{11}{(x+1)^2}.$$

22. Find 2 positive numbers whose sum is 400 and have the largest possible product.

Solution: $x_{max} = 200$ and $y_{max} = 200$.

Let x and y be the positive numbers.

$x + y = 400 \Rightarrow y = 400 - x$

$P = xy = x(400 - x) = 400x - x^2$

The maximum product occurs if $P' = 0$.

$P' = 400 - 2x_{max} = 0 \Rightarrow x_{max} = 200$

$y_{max} = 400 - x_{max} \Rightarrow y_{max} = 400 - 200$

$y_{max} = 200$.

23. Graph the function $y = x^3 - 9x^2$.

a) Find the derivatives y' and y''.
b) Find x-y intercepts.
c) Find the maximum and minimum points.
d) Find the inflection point.
e) Graph the function y.

Solution:

a) Find the derivatives y' and y''.

$y = x^3 - 9x^2$

$y' = 3x^2 - 18x$

$y'' = 6x - 18$

b) Find x-y intercepts.

x-intercept $(y = 0)$

$y = x^3 - 9x^2 \Rightarrow x^2(x - 9) = 0 \Rightarrow x_{int} = 0$ and $x_{int} = 9$

y-intercept $(x = 0)$

$y = x^3 - 9x^2 \Rightarrow y = (0)^3 - 9(0)^2 \Rightarrow y_{int} = 0$

c) Find the maximum and minimum points.

$y' = 0 \Rightarrow 3x^2 - 18x = 0 \Rightarrow 3x(x - 6) = 0$

$x_{max} = 0 \; (y''(0) = -18 < 0) \Rightarrow P_{max}(0,0)$.

$x_{min} = 6 \; (y''(4) = 18 > 0) \Rightarrow P_{min}(6, -108)$.

d) Find the inflection point.

$y'' = 0 \Rightarrow 6x - 18 = 0 \Rightarrow x_{inf} = 3 \Rightarrow P_{inf}(3, -54)$.

e) Graph the function y.